Roll No. ....

D–990

M. A./M. Sc. (Fourth Semester) (Main/ATKT)

EXAMINATION, May-June, 2020

MATHEMATICS

Paper Fourth (B)

(Wavelets—II)

Time : Three Hours] [*Maximum Marks : 80*

Note : Attempt all Sections as directed.

Section—A 1 each

(Objective/Multiple Choice Questions)

Note : Attempt all questions.

Choose the correct answer :

1. If $\psi \in L^2(\mathbf{R})$, then expression for $(\psi_{j,k})\,\hat{}\,(\xi)$ is :

   (a) $2^{j/2}\psi(2^j x - k)$

   (b) $2^{-j/2}\hat{\psi}(2^{-j}\xi)\, e^{i\,2^{-j}k\xi}$

   (c) $2^{-j/2}\hat{\psi}(\xi)\, e^{i\,2^{-j}k\xi}$

   (d) None of the above

2. If $\psi \in L^2(\mathbf{R})$, then which of the following is true ?

   (a) $\sum_{j,k\in\mathbf{Z}} \left|< f, \psi_{j,k} >\right|^2 \leq \|f\|_2^2$

   (b) $\sum_{j,k\in\mathbf{Z}} \left|< f, \psi_{j,k} >\right|^2 \geq \|f\|_2^2$

   (c) $\sum_{j,k\in\mathbf{Z}} \left|< f, \psi_{j,k} >\right|^2 = \|f\|_2^2$

   (d) None of the above

3. If $\{e_j : j = 1, 2, ...\}$ is a system of vectors in a Hilbert space H, satisfying :

$$\|f\|_2^2 = \sum_{j=1}^{\infty} \left|< f, e_j >\right|^2$$

   then for $\{e_j : j = 1, 2, ...\}$ to be $j = 1$ a orthnormal basis condition is :

   (a) $\|e_j\| \geq 1$ for $j = 1, 2, ...$

   (b) $\|e_j\| = 1$ for $j = 1, 2, ...$

   (c) $\|e_j\| \leq 1$ for $j = 1, 2, ...$

   (d) None of the above

4. If $\{e_j : j = 1, 2, ...\}$ is a system of vectors in H, then the expression for Nth partial sum for $f \in H$ is :

   (a) $S_N = \sum_{j=1}^{\infty} < f, e_j > e_j$

   (b) $S_N = \sum_{j=0}^{N} < f, e_j > e_j$

   (c) $S_N = \sum_{j=1}^{N} < f, e_j > e_j$

   (d) None of the above

5. Give an example of dense subset of $L^2(\mathbf{R})$.

6. Define $l^2(z)$.

7. If ψ is an MRA wavelet, then :

   $$\dim F_{\psi}(\xi) = ?$$

   for $\xi \in \mathbf{T}$:

   (a) 1

   (b) 0

   (c) ∞

   (d) None of the above

8. Limit of a sequence of MRA wavelet is a .... wavelet.

   (a) Band limited

   (b) Convergent

   (c) Continuous

   (d) None of the above

9. Define low pass filter.

10. Define $V_0^{\mathbf{I}}$.

11. Every orthonormal wavelet is a frame. (True/False)

12. Domain and codomain of a frame operator F are respectively :

    (a) H and $l^2(J)$

    (b) $l^2(\mathbf{R})$ and H

    (c) $l^2(\mathbf{R})$ and $l^2(J)$

    (d) None of the above

13. Frame bounds for Zak transform $\mathbf{R}_g(s, t)$ are given by :

    (a) $0 < A \le |R_g(s, t)| \le B. < \infty$

    (b) $-A < |R_g(s, t)| \le +B < \infty$

    (c) $0 < A \le |R_g(s, t)|^2 \le B < \infty$

    (d) None of the above

14. Domain and codomain of a Zak transform R are given by :

    (a) $L^2(\mathbf{T}^2)$ and $L^2(\mathbf{R})$ respectively.

    (b) $L^2(\mathbf{R})$ and $L^2(\mathbf{T}^2)$ respectively.

    (c) $L^2(\mathbf{R}^2)$ and $L^2(\mathbf{T})$ respectively.

    (d) None of the above

15. Define $H^2(\mathbf{R})$.

16. An expression of discrete cosine bases $u_{j,k}(x) = ?$

(a) $u_{j,k}(x) = \sqrt{\frac{2}{l_j}}\, w_j(x) \cos\left(\pi\left(k + \frac{1}{2}\right)\left(\frac{x - a_j}{l_j}\right)\right)$

(b) $u_{j,k}(x) = \sqrt{\frac{2}{l_j}}\, w_j(x)$

(c) $u_{j,k}(x) = \sqrt{\frac{2}{w_j(x)}}\, l_j(x) \cos(\pi k x)$

(d) None of the above

17. The discrete version for local sine and cosine bases support of $F_j = ?$, where .... F_j is a subspace of $l^2(\mathbf{Z})$.

18. The expression for filter $m_1(\xi)$ used in the decomposition algorithm of wavelets is :

(a) $\sum_{n \in \mathbf{Z}} \alpha_n e^{in\xi}$

(b) $e^{i\xi}\overline{m_0(\xi)}$

(c) $e^{i\xi}\overline{m_0(\xi + \pi)}$

(d) None of the above

19. An expression for $\phi_{j-1,k}(x)$, which belongs to V_{j-1} is :

(a) $\sqrt{2}\sum_{n \in \mathbf{Z}} \alpha_n\, \phi_{j,k}(x)$

(b) $\sqrt{2}\sum_{n \in \mathbf{Z}} \alpha_n\, \phi_{j,2k-n}(x)$

(c) $\sqrt{2}\sum_{n \in \mathbf{Z}} \alpha_n\, \psi_{j,2k-n}(x)$

(d) None of the above

20. An expression for $d_{j-1,\,k}$ in the decomposition algorithm of Haar wavelet is :

(a) $d_{j-1,\,k} = \sqrt{2}\left[\frac{C_{j,2k-1} - C_{j,2k}}{2}\right]$

(b) $d_{j-1,\,k} = \sqrt{2}\, C_{j,2k-1}$

(c) $d_{j-1,\,k} = \sqrt{2}\left[\frac{C_{j,k-1} - C_{j,k}}{2}\right]$

(d) None of the above

Section—B 2 each

(Very Short Answer Type Questions)

Note : Attempt all questions.

1. Write two basic equations for $\psi \in L^2(\mathbf{R})$ to be an orthonormal wavelet.
2. Define M. S. F. wavelet.
3. Write necessary and sufficient conditions for a function $\phi \in L^2(\mathbf{R})$ to be a scaling function.
4. If ψ is an orthonormal wavelet and :

$$G_n(\xi) = \sum_{j=1}^{\infty}\sum_{k \in \mathbf{Z}} \hat{\psi}\left(2^n(\xi + 2k\pi)\right)\overline{\hat{\psi}\left(2^j(\xi + 2k\pi)\right)}\,\hat{\psi}(2^j\xi)$$

a.e., then show that :

$$G_n(\xi) = G_{n-1}(2\xi)$$

5. Define dual frame.
6. Give the statement of Balian low theorem for frames.
7. Define window w_i for discrete version of local sine and cosine transform.
8. Define projection $P_j f(x)$ from $L^2(\mathbf{R})$ onto V_j.

Section—C 3 each

(Short Answer Type Questions)

Note : Attempt any *eight* questions.

1. Suppose that :

$$\{e_j : j = 1, 2, ...\}$$

is a system of vectors in a Hilbert space **H** satisfying :

$$\|f\|^2 = \sum_{j=1}^{\infty} |< f, e_j >|^2$$

for all $f \in \mathbf{H}$. If $\|e_j\| \geq 1$ for j = 1, 2, ... ; then prove that $\{e_j\}$ is an orthonormal basis for **H**.

2. If ψ is an orthonormal wavelet and $|\hat{\psi}|$ is continuous at zero, then prove that :

$$\hat{\psi}(0) = 0$$

3. Suppose that :

$$\{e_j : j = 1, 2, ...\}$$

is a family of elements in a Hilbert space H such that equality :

$$\|f\|^2 = \sum_{j=1}^{\infty} |< f, e_j >|^2$$

holds all f belongs to a dense subset D of H, then prove that the equality is valid for all $f \in$ H.

4. Let :

$$\mu_1, \mu_2, ..., \mu_n$$

be 2π-periodic functions and set :

$$M_j = \sup_{\xi \in \mathbf{T}} \left(|\mu_j(\xi)|^2 + |\mu_j(\xi + \pi)|^2 \right)$$

then prove that :

$$\int_{-2^n \pi}^{2^n \pi} \prod_{j=1}^{n} |\mu_j(2^{-j}\xi)|^2 \, d\xi \leq 2\pi \, \mu_1, ... \mu_n$$

5. Suppose that :

$$\{\psi^{(n)} : n = 1, 2, ...\}$$

is a sequence of MRA wavelet converging to ψ in $L^2(\mathbf{R})$. If ψ is also a wavelet, then prove that ψ must be an MRA wavelet.

6. Let :

$$H = C^2$$

and $\phi_1 = (0, 1)$

$$\phi_2 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\phi_3 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

then find the values of A and B for $\{\phi_j\}_{j=1,2,3}$ to be a frame for **H**.

7. If :

$$g \in L^2(\mathbf{R})$$

$$(Q g)(x) = x\, g(x)$$

and $$(Pg)(x) = -i\, g'(x)$$

prove that :

$$< Qg, \tilde{g}_{m,n} > = < g_{-m,-n}, Q\tilde{g} >$$

and $$< Pg, \tilde{g}_{m,n} > = < g_{-m,-n}, P\tilde{g} >$$

8. Prove that :

$$\sum_{k=0}^{N-1} \cos\left(\pi\left(x+\frac{1}{2}\right)\frac{M}{N}\right) = 0 \quad \text{for } 1 \le M \le 2N-1.$$

9. Explain in short how Haar wavelet works for decomposition algorithm.

Section—D 5 each

(Long Answer Type Questions)

Note : Attempt all questions.

1. Let $\psi \in L^2(\mathbf{R})$ be such that $|\hat{\psi}| = \chi_k$ for a measurable set $k \subset \mathbf{R}$. Then prove that ψ is a wavelet if and only if there exist a partition $\{I_l : l \in \mathbf{Z}\}$ of I, partition $\{k_l : l \in \mathbf{Z}\}$ of k, and two integer valued sequences $\{j_l ; l \in \mathbf{Z}\}$, $\{k_l : l \in \mathbf{Z}\}$ such that :

(i) $k_l = 2^{j_l} I_l \; l \in \mathbf{Z}$

(ii) $\{k_{l+2k_l\pi} : l \in \mathbf{Z}\}$

is a partition of I.

Or

Let **H** be a Hilbert space and $\{e_j : j = 1, 2, ...\}$ be a family of elements of **H**. Then prove that :

(i) $\|f\|^2 = \sum_{j=1}^{\infty} |< f, e_j >|^2$ holds for all $f \in \mathbf{H}$ if and only if

(ii) $f = \sum_{j=1}^{\infty} < f, e_j > e_j$ with convergence in H, for all $f \in H$.

2. Let $\{v_j : j \ge 1\}$ be a family of vectors in a Hilbert space **H** such that :

(i) $\sum_{n=1}^{\infty} \|v_n\|^2 = C < \infty$

(ii) $v_n = \sum_{m=1}^{\infty} < v_n , v_m > v_n$ for all $n \geq 1$.

Let :

$$F = \overline{\text{span}\left\{v_j : j \geq 1\right\}}$$

Then prove that :

$$\dim F = \sum_{j=1}^{\infty} \left\| v_j \right\|^2 = C .$$

Or

Let :

$$m_0 \in C'(\mathbf{R})$$

be a 2π-periodic function which satisfies :

$$M_0 \in 1$$

$$\left|m_0(\xi)\right|^2 + \left|m_0(\xi + \pi)\right|^2 = 1$$

for $\xi \in \mathbf{R}$ and there exists a set $k \subset R$ which is a finite union of closed bounded intervals such that 0 is in the interior of k :

$$\sum_{k \in \mathbf{Z}} \chi_k(\xi + 2k\pi) = 1 \text{ for } \xi \in \mathbf{R}$$

and $m_0(2^{-j}\xi) \neq 0$ for all j = 1, 2, ... and all $\xi \in k$. Then prove that m_0 is the low pass filter for an MRA.

3. Suppose :

$$g \in L^2(\mathbf{R})$$

and
$$\left\{ \underset{m,\ n}{g}\ (x) = e^{2\pi inx} g\ (x - n) : \right\}$$

is a frame for $L^2(\mathbf{R})$, S = F*F, with $F'^{m,n \in \mathbf{Z}}$ is a frame operator. Prove that F*F commutes with translation by integers with integer modulation.

Or

Suppose that :

$$\{e_j : j = 1, 2, ...\}$$

is a family of elements in a Hilbert space **H** such that there exist constants $0 < A \leq B < \infty$ satisfying :

$$A \left\| f \right\|^2 \leq \sum_{j=1}^{\infty} \left| < f, e_j > \right|^2 \leq B \left\| f \right\|^2$$

for all f belonging to a dense subset D of **H**. Then prove that the same inequalities are true for all $f \in \mathbf{H}$.

4. Explain in details that what do you mean by reconstruction algorithm for wavelets.

Or

Prove that the sequence :

$$\left\{u_{j,k} : j \in \mathbf{Z}, 0 \leq k \leq l_j - 1\right\}$$

is an orthonormal basis for $l^2(\mathbf{Z})$.